

TEST ELEMENTS IN TORSION-FREE HYPERBOLIC GROUPS

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ABSTRACT. We prove that in a torsion-free hyperbolic group, an element is a test element if and only if it is not contained in a proper retract.

Definition 1. [12, Definition 1], [6, Definition 1] Let G be a group. An element $g \in G$ is a *test element* if any endomorphism $\phi: G \rightarrow G$ for which $\phi(g) = g$ is an automorphism of G .

Nielsen [5] proved that $[a, b]$ is a test element in $F_2 = \langle a, b \mid \rangle$. Other test elements were found by Zieschang [13] and Shpilrain [11].

Examples 2. Suppose that F_r is a free group of rank r , with basis $\{a_1, \dots, a_r\}$. For $k \geq 2$, the element $a_1^k \cdots a_r^k$ is a test element. If r is even, the element $[a_1, a_2] \cdots [a_{r-1}, a_r]$ is a test element. See [13, 12].

Clearly, if $g \in G$ is contained in a proper retract of G , then g cannot be a test element.

Definition 3. [6, Definition 2] A hyperbolic group G is *stably hyperbolic* if for every endomorphism $\phi: G \rightarrow G$, there are arbitrarily large values of n so that $\phi^n(G)$ is hyperbolic.

O'Neill and Turner [6] proved the following result.

Theorem 4. [6, Theorem 1] *Suppose that G is a torsion-free and stably hyperbolic group. Then $g \in G$ is a test element if and only if g is not contained in a proper retract of G .*

We do not know if every torsion-free hyperbolic group is stably hyperbolic, as conjectured by O'Neill and Turner. However, we can prove that the above retract theorem holds for all torsion-free hyperbolic groups.

Theorem 5. *Suppose that G is a torsion-free hyperbolic group. An element $g \in G$ is a test element if and only if g is not contained in a proper retract of G .*

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The proof of this theorem uses Sela's Shortening Argument, and the theory of JSJ decompositions of groups. We attempt to give references, though everything we do is standard in this area, and we assume the reader is familiar with these techniques. For an introduction the general theory of JSJ decompositions, see [3, 4].

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1. PROOF OF THEOREM 5

Throughout, G is a torsion-free hyperbolic group, $\phi: G \rightarrow G$ is an endomorphism and $g \in G$ satisfies $\phi(g) = g$. First note that according to the main result of [8], if ϕ is surjective then it is an automorphism. Also, we have the following result.

Theorem 6. *[Sela] There exists an $N \in \mathbb{N}$ so that for all $n \geq N$ we have*

$$\ker(\phi^n) = \ker(\phi^N).$$

Remark 7. Theorem 6 is claimed in [8] (it does not require ϕ to fix any element of G), though a proof does not appear there. However, if G is a torsion-free hyperbolic group, then G and its endomorphic images are all G -limit groups, in the sense of [10, Definition 1.11]. Thus, Theorem 6 is an immediate consequence of [10, Theorem 1.12], the descending chain condition for G -limit groups.

The sequence of kernels $\ker(\phi^i)$ is an ascending chain of subgroups of G . Theorem 6 says that this sequence stabilizes. In particular

$$\phi^i \big|_{\phi^N(G)}$$

is injective for all $i \geq 1$.

Consider the group $H = \phi^N(G)$, as an abstract finitely generated group. Clearly, if we choose a different value of N , still satisfying the conclusion of Theorem 6, the group H is unchanged (as an abstract group).

Let $\pi: H \rightarrow \phi^N(G)$ be an isomorphism, and let $g_\pi = \pi^{-1}(g)$.

Observation 8. *Since roots are unique in torsion-free hyperbolic groups, if $C_G(g) = \langle \gamma \rangle$, then $\phi(\gamma) = \gamma$, and $\gamma \in \phi^i(G)$ for any i . Therefore, we suppose henceforth that g generates its own centralizer (so that it is not a proper power in G). Thus we may assume that g_π is not a proper power in H .*

Definition 9. Let Γ be a group and Λ a subgroup of Γ . We say that Γ is *freely indecomposable rel Λ* if there is no proper free product decomposition $\Gamma = \Gamma_1 * \Gamma_2$ where $\Lambda \leq \Gamma_1$.

The relative version of Grushko's Theorem is the result below. The proof is the same as the usual version of Grushko's Theorem, except that only free splittings where Λ is contained in one factor are considered. See [3, §4.2] for a discussion about why JSJ decompositions (including the Grushko decomposition) can be performed in the relative case. The following statement can also be found in [1].

Theorem 10. *Let Γ be a finitely generated group and Λ a subgroup of Γ . There is a free product decomposition*

$$\Gamma = \Gamma_\Lambda * \Gamma_1 * \cdots * \Gamma_k * F$$

where

- (1) $\Lambda \leq \Gamma_\Lambda$;
- (2) Γ_Λ is freely indecomposable rel Λ ;
- (3) The Γ_i are freely indecomposable and not free; and
- (4) F is a finitely generated free group.

The subgroup Γ_Λ is unique. Up to reordering and conjugation, the Γ_i are unique. The rank of F is determined by Γ, Λ . This splitting is called the Grushko decomposition of Γ rel Λ .

Consider the Grushko decomposition of H rel C , where $C = \langle g_\pi \rangle$. The subgroup H_C is freely indecomposable rel C and is a retract of H .

Whenever Γ is a finitely generated group and Λ is a subgroup, so that Γ is freely indecomposable rel Λ , there is a relative cyclic JSJ decomposition of Γ rel Λ . This has the form of a graph of groups with cyclic edge groups. There is a distinguished vertex group V_Λ , which contains Λ . Other vertices are either cyclic, *QH-subgroups*, which are isomorphic to the fundamental group of a 2-orbifold with boundary so that the adjacent edge groups correspond to boundary components or are *rigid* (which just means they are not of the first two types).

That the cyclic JSJ decomposition of H_C rel C exists follows as in the paragraph at the end of [10, §1]¹. For an alternative explanation, note that since H_C is a subgroup of a torsion-free hyperbolic group, it is torsion-free and CSA. Therefore, the existence of the required splitting follows from [3, Theorem 11.1].

Let $\mathcal{T}(H_C, C)$ be the canonical cyclic JSJ decomposition of H_C rel C . Let V_C be the distinguished vertex containing C .

The *modular group* of H_C rel C , denoted $\text{Mod}(H_C, C)$ may be defined to be the group of automorphisms of H_C generated by (i) inner automorphisms of H_C fixing g_π (these are conjugation by powers of g_π); (ii) Dehn twists in edge groups of $\mathcal{T}(H_C, C)$; and (iii) Dehn twists

¹This argument in turn follows that in [9, §9]

in essential simple closed curves in surfaces corresponding to QH subgroups of $\mathcal{T}(H_C, C)$. By convention, we choose Dehn twists which fix V_C element-wise.

Suppose that $X(H_C, C) = \{\eta: H_C \rightarrow G \mid \eta \text{ injective, } \eta(g_\pi) = g\}$.

There is a natural action by pre-composition of $\text{Mod}(H_C, C)$ on $X(H_C, C)$. The Shortening Argument implies the following:

Theorem 11. *The set $X(H_C, C)/\text{Mod}(H_C, C)$ is finite.*

Theorem 11 follows from the construction of the restricted Makanin-Razborov diagram for H_C as in [10, §1] (see also [9, §8] for more details in the similar situation of a free group). This diagram encodes all of the homomorphisms from H_C to G , where we force certain elements to have given image. There are proper quotients of H_C in this diagram, but we are only considering injective homomorphisms, so we are only concerned about the end of the diagram, which consists of finitely generated subgroups of G along with injective homomorphisms into G . Theorem 11 is just a restatement about this last part of the restricted Makanin-Razborov diagram.

Note that normally one might expect to have to shorten by inner automorphisms of G , but in this case we are fixing the image of g_π , so we can only conjugate by elements centralizing g , and this can be achieved by inner automorphisms of H_C . The limiting \mathbb{R} -tree in this construction is described in detail in the proof of Proposition 3.6 in [2].

The Main Theorem is a fairly easy consequence of Theorem 11, as follows.

Suppose that $\psi_0 = \phi^N$, so that $\psi_0(G) \cong H$, and recall that $\pi: H \rightarrow \psi_0(G)$ is an isomorphism. Note that $\psi_0|_{\psi_0(G)}$ is injective. Let $\eta: H \rightarrow H_C$ be the canonical retraction and $\iota: H_C \rightarrow H$ be the inclusion, so that $\eta \circ \iota = \text{Id}_{H_C}$. Let $K = \pi^{-1}(H_C)$.

We have a homomorphism $\kappa: G \rightarrow K$ defined by

$$\kappa = \pi^{-1} \circ \iota \circ \eta \circ \pi \circ \psi_0.$$

We note that $\kappa(g) = \pi^{-1}(\iota(\eta(\pi(g)))) = g$, and that $\kappa|_K$ is injective, since $\iota \circ \eta|_{H_C}$ is injective and π is an isomorphism.

For a positive integer s , define a homomorphism $\xi_s: H_C \rightarrow G$ by

$$\xi_s = \kappa^s \circ \pi.$$

The above observations show that we have $\xi_s \in X(H_C, C)$ for any $s \geq 1$.

By Theorem 11 there are positive integers k, j with $k > j$ and $\alpha \in \text{Mod}(H_C, C)$ so that

$$\xi_k = \xi_j \circ \alpha.$$

Let $\beta = \pi \circ \alpha \circ \pi^{-1}$ be the automorphism of K induced by α . When all homomorphisms in the next equation are restricted to have K as domain, we have

$$\kappa^k = \xi_k \circ \pi^{-1} = \xi_j \circ \alpha \circ \pi^{-1} = \kappa^j \circ \pi \circ \alpha \circ \pi^{-1} = \kappa^j \circ \beta.$$

Now, κ is injective on K , so we have $\kappa^{k-j}|_K = \beta$, so $\kappa^{k-j}(K) = K$, and $\beta^{-1} \circ \kappa^{k-j}$ is the identity map on K .

Therefore, $\beta^{-1} \circ \kappa^{k-j}: G \rightarrow K$ is a retraction and $g \in K$. If ϕ is not an automorphism then we know that it is not surjective. Since $K \leq \phi^N(G)$, in this case we clearly have $K \neq G$, so it is a proper retract. This completes the proof of Theorem 5.

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